AN EXAMPLE OF ONE - DIMENSIONAL PHASE TRANSITION

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Abstract. In the paper a one-dimensional model with nearest - neighbor interactions $I_n, n \in \mathbf{Z}$ and spin values ± 1 is considered. We describe a condition on parametres I_n under which the phase transition occurs. In particular, we show that the phase transition occurs if $I_n \geq |n|, n \in \mathbf{Z}$.

Keywords: Configuration; One-dimension; Phase transition; Gibbs measure

1 Introduction

Interest in phase transition in one-dimensional systems has gained due to Little's work [8]. Existence or nonexistence depends markedly upon the model employed. In [11] Van Hove shows (see also [9, section 5.6.]) a one-dimensional system can not exhibit a phase transition if the (translation-invariant) forces are of finite range.

An interesting one-dimensional model considered by Baur and Nosanow [1], giving rise to a phase transition with only nearest- neighbor interactions which has some of the interaction constants equal to minus infinity. Note that in a particular case the model which we shall consider here is a model with only nearest-neighbor interactions without assumption of existence a interaction equal to minus infinity (cf. with [1]).

An other examples of one-dimensional phase transitions for models with long range interactions were considered in [3-7], [10].

In the paper we consider the Hamiltonian

$$H(\sigma) = \sum_{l=(x-1,x):x\in\mathbf{Z}} I_x \mathbf{1}_{\sigma(x-1)\neq\sigma(x)}, \tag{1}$$

where $\mathbf{Z} = \{..., -1, 0, 1, ...\}$, $\sigma = \{\sigma(x) \in \{-1, 1\} : x \in \mathbf{Z}\} \in \emptyset = \{-1, 1\}^{\mathbf{Z}}$, and $I_x \in R$ for any $x \in \mathbf{Z}$.

The goal of this paper is to describe a condition on parameters of the model (1) under which the phase transition occurs.

2 Phase transition

Let us consider the sequence $Ł_n = [-n, n], n = 0, 1, ...$ and denote $Ł_n^c = \mathbf{Z} \setminus Ł_n$. Consider boundary condition $\sigma_n^{(+)} = \sigma_{Ł_n^c} = \{\sigma(x) = +1 : x \in Ł_n^c\}$. The energy $H_n^+(\sigma)$ of the

configuration σ in the presence of boundary condition $\sigma_n^{(+)}$ is expressed by the formula

$$H_n^+(\sigma) = \sum_{l=(x-1,x):x\in Ł_n} I_x \mathbf{1}_{\sigma(x-1)\neq\sigma(x)} + I_{-n}\mathbf{1}_{\sigma(-n)\neq 1} + I_{n+1}\mathbf{1}_{\sigma(n)\neq 1}. \tag{2}$$

The Gibbs measure on $\emptyset_n = \{-1,1\}^{Ł_n}$ with boundary condition $\sigma_n^{(+)}$ is defined in the usual way

$$\mu_{n,\beta}^+(\sigma) = Z^{-1}(n,\beta,+)\exp(-\beta H_n^+(\sigma)), \tag{3}$$

where $\beta = T^{-1}$, $T > 0-$ temperature and $Z(n,\beta,+)$ is the normalizing factor (statistical sum).

Denote by σ_n^+ the configuration on $\mathbf{Z}$ such that $\sigma_n^+(x) \equiv +1$ for any $x \in Ł_n^c$.

Put

$$A(\sigma_n^+) = \{x \in \mathbf{Z} : \sigma_n^+(x) = -1\},$$

$$\partial(\sigma_n^+) = \{l = (m-1,m) \in \mathbf{Z}\times\mathbf{Z} : \sigma_n^+(m-1) \neq \sigma_n^+(m)\}.$$

Note that there is one-to-one correspondence between the set of all configurations σ_n^+ and the set of all subsets A of $Ł_n$.

Let $A'(\sigma_n^+)$ be the set of all maximal connected subsets of $A(\sigma_n^+)$.

LEMMA 1. *Let $B \subset \mathbf{Z}$ be a fixed connected set and $p_\beta^+(B) = \mu_{n,\beta}^+\{\sigma_n^+ : B \in A'(\sigma_n^+)\}$. Then*

$$p_\beta^+(B) \le \exp\Big\{ -\beta\Big[I_{n_B} + I_{N_B+1}\Big]\Big\},$$

where n_B (resp. N_B) is the left (resp. right) endpoint of B.

Proof. Denote $F_B = \{\sigma_n^+ : B \in A'(\sigma_n^+)\}-$ the set of all configurations σ_n^+ on Z with "+"-boundary condition (i.e. $\sigma_n^+(x) \equiv +1$ for any $x \in Ł_n^c$) such that B is maximal connected set. Denote also $F_B^- = \{\sigma_n^+ : B \cap A'(\sigma_n^+) = \emptyset\}$. Define the map $\chi_B : F_B \to F_B^-$ as following: for $\sigma_n \in F_B$ we destroy the set B changing the values $\sigma_n(x)$ inside of B to $+1$. The constructed configuration is $\chi_B(\sigma_n) \in F_B^-$.

For a given B the map χ_B is one-to-one map.

For $\sigma_n \in F_B$ it is clear that

$$A'(\sigma_n) = A'(\chi_B(\sigma_n)) \cup B, \quad \partial(\sigma_n) = \partial(\chi_B(\sigma_n)) \cup \{(n_B-1,n_B),(N_B,N_B+1)\}.$$

Thus we have

$$H_n^+(\sigma_n) - H_n^+(\chi_B(\sigma_n)) = I_{n_B} + I_{N_B+1}. \tag{4}$$

By definition we have

$$p_\beta^+(B) = \frac{\sum\limits_{\sigma_n\in F_B} \exp\{-\beta H_n^+(\sigma_n)\}}{\sum\limits_{\sigma_n} \exp\{-\beta H_n^+(\sigma_n)\}} \le \frac{\sum\limits_{\sigma_n\in F_B} \exp\{-\beta H_n^+(\sigma_n)\}}{\sum\limits_{\sigma_n\in F_B} \exp\{-\beta H_n^+(\chi_B(\sigma_n))\}}. \tag{5}$$

Using (4) from (5) we get

$$p_\beta^+(B) \leq \frac{\sum\limits_{\sigma_n\in F_B} \exp\Big\{-\beta H_n^+(\chi_B(\sigma_n)) - \beta\Big[I_{n_B} + I_{N_B+1}\Big]\Big\}}{\sum\limits_{\sigma_n\in F_B} \exp\{-\beta H_n^+(\chi_B(\sigma_n))\}} =$$

$$\exp\Big\{-\beta\Big[I_{n_B} + I_{N_B+1}\Big]\Big\}.$$

The lemma is proved.

Assume that for any $r \in \{1, 2, ...\}$ and $n \in \mathbf{Z}$ the Hamiltonian (1) satisfies the following condition

$$I_n + I_{n+r} \geq r. \tag{6}$$

LEMMA 2. *Assume condition (6) is satisfied. Then for all sufficiently large β, there is a constant $C = C(\beta) > 0$, such that*

$$\mu_\beta^+\{\sigma_n : \quad |B| > C \ln |Ł_n| \quad \textit{for some} \quad B \in A'(\sigma_n)\} \to 0, \quad \textit{as} \quad |Ł_n| \to \infty,$$

where $|\cdot|$ denotes the number of elements.

Proof. Suppose $\beta > 1$, then by Lemma 1 and condition (6) we have

$$\mu_\beta^+\{\sigma_n : B \in A'(\sigma_n), \quad t \in B, |B| = r\} = \sum_{B:\quad t\in B, \quad |B|=r} p_\beta^+(B) \leq r \exp\{-\beta r\}.$$

Hence

$$\mu_\beta^+\{\sigma_n : B \in A'(\sigma_n), \quad t \in B, |B| > C_1 \ln |Ł_n|\} \leq \sum_{r\geq C_1 \ln |Ł_n|} r \exp\{-\beta r\} \leq$$

$$\sum_{r\geq C_1 \ln |Ł_n|} \exp\{(1-\beta)r\} = \frac{|Ł_n|^{C_1(1-\beta)}}{1-e^{1-\beta}}, \tag{7}$$

where C_1 will be defined latter. Thus we have

$$\mu_\beta^+\{\sigma_n : \exists B \in A'(\sigma_n), \quad |B| > C_1 \ln |Ł_n|\} \leq \frac{|Ł_n|^{C_1(1-\beta)+1}}{1-e^{1-\beta}}.$$

The last expression tends to zero if $|Ł_n| \to \infty$ and $C_1 > \frac{1}{\beta-1}$. The lemma is proved.

LEMMA 3. *Assume the condition (6) is satisfied. Then*

$$\mu_\beta^+\{\sigma_n : \sigma_n(0) = -1\} \to 0, \quad \textit{as} \quad \beta \to \infty. \tag{8}$$

Proof. If $\sigma_n(0) = -1$, then 0 is point for some $B \in A'(\sigma_n)$. Consequently,

$$\mu_\beta^+\{\sigma_n : 0 \in B, \quad |B| < C_1 \ln |Ł_n|\} \leq \sum_{r=1}^{C_1 \ln |Ł_n|} (e^{1-\beta})^r \leq \frac{e^{1-\beta}}{1-e^{1-\beta}}$$

and

$$\mu_\beta^+\{\sigma_n(0)=-1\} \leq \mu_\beta^+\{\sigma_n : 0\in B, \quad B\in A'(\sigma_n)\} \leq$$
$$\frac{e^{1-\beta}}{1-e^{1-\beta}} + \frac{|\text{Ł}_n|^{C_1(1-\beta)+1}}{1-e^{1-\beta}}. \tag{9}$$

For $|\text{Ł}_n| \to \infty$ and $\beta \to \infty$ from (9) we get (8). The lemma is proved.

THEOREM 4. *Assume the condition (6) is satisfied. For all sufficiently large β there are at least two Gibbs measures for the model (1) .*

Proof. Using a similar argument one can prove

$$\mu_\beta^-\{\sigma_n : \sigma_n(0)=1\} \to 0, \quad \text{as} \quad \beta \to \infty.$$

Consequently, for sufficiently large β we have

$$\mu_\beta^+\{\sigma_n : \sigma_n(0)=-1\} \neq \mu_\beta^-\{\sigma_n : \sigma_n(0)=-1\}.$$

This completes the proof.

Denote

$$\mathcal{H} = \{H : H \text{ (see (1)) satisfies the condition (6)}\}$$

The following example shows that the set $\mathcal{H}$ is not empty.

Example. Consider Hamiltonian (1) with $I_m \geq |m|, \quad m \in \mathbf{Z}$. Then

$$I_m + I_{m+k} \geq |m| + |m+k| \geq k$$

for all $m \in \mathbf{Z}$ and $k \geq 1$. Thus the condition (6) is satisfied.

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